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UTILIZATION OF SHIFTED LEGENDRE AND TOUCHARD POLYNOMIALS AS BASIS FUNCTIONS FOR COMPARATIVE NUMERICAL INVESTIGATION OF SECOND-KIND LINEAR VOLTERRA EQUATIONS

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ABSTRACT

In this paper, shifted Legendre and Touchard polynomials are utilized as trial functions at different values for the collocation approach to solve second-kind linear Volterra integro-differential equations. The residual is produced by replacing the original problem with the basis function, which is then collocated at predetermined locations inside the problem domain. A set of linear algebraic equations are produced by applying the boundary conditions to the basis functions. There are, three numerical experiments, employed to investigate the Volterra integro-differential equations. Both tables and graphics are used to present the results. Calculating the error associated with each problem set revealed that the Touchard polynomial converges more quickly than the shifted Legendre polynomial in all of the numerical cases under consideration. It is recommended that the collocation approach be used in conjunction with either of shifted Legendre or Touchard polynomials as a basis function for the solution of linear Volterra integro-differential equations due to their high degrees of accuracies obtained from the problems studied in this research.

Keywords: *Shifted Legendre Polynomial, Touchard Polynomial, Collocation Method, Second-kind Volterra Equation, Approximate solutions.*

1.0 INTRODUCTION

According to recent findings, applications of integro-differential equation has gained more attention in physics, engineering, biology, and control theory due to their ability to describe processes with memory and hereditary effects. It is one kind of functional equation that has both an integral and the derivatives of an unknown function. Integro-differential equations (IDEs) are created by combining differential and integral equations. Examining IDEs does not make the challenges that come up when studying differential and integral equations any easier. Also, many branches of applied mathematics, engineering, and mathematical physics naturally use IDEs as observed by Aggarwal *et al.* (2016). To this purpose, a number of approaches to solving this class of problems have been put out in the literature. For instance, Arqub *et al.* (2009) used the method of reproducing kernel to solve Volterra integro-differential equations.

Furthermore, Berenguer *et al.* (2012) introduced a solution method based on the fixed point iterative algorithm while Maleknejad *et al.* (2012) presented a Bernstein operational-matrix-based method for the solution of Volterra integro-differential equations. In Maleknejad and Attary (2012), a Chebyshev collocation points together with the Shannon approximation was introduced to transform Volterra integro-differential equations system to an algebraic system. Ibrahim and Ayoo (2015) used a new iterative method to solve integro-differential equations which yielded accurate results. These techniques often convert the Volterra integro-differential equations into a set of equations that can be resolved directly or by iteration.

Numerical algorithms are used in numerical approximation techniques to tackle discrete problems and produce approximate solutions. Depending on how complicated the problem is and how accurate the answer must be, these algorithms can be anything from straightforward iterative processes to intricate optimization strategies. Numerical approximation techniques require error analysis. It entails measuring how much the actual answer to the mathematical issue differs from the approximate solution that

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was determined numerically. The selection of suitable approximation methods and parameters is guided by error analysis, which also aids in evaluating the precision and dependability of the numerical solution. A numerical solution is said to be convergent as its discretization parameters (such grid size or time step) get closer to zero (Wazwaz, 2015).

Shifted Legendre polynomials are a kind of orthogonal polynomials that appear in numerical techniques and mathematical analysis. They share many similarities with the standard Legendre polynomials, but instead of being specified on the standard interval [0,1], they are defined on a shifted interval of [−1,1] (Cao *et al.*, 2020). Significant uses for shifted Legendre polynomials can be found in numerical integration, approximation theory, and differential equation's solving. According to Jena *et al.* (2020), shifted Legendre polynomials are computationally efficient to evaluate and work with since they share recurrence connections and precise formulas for their coefficients with standard Legendre polynomials.

Touchard polynomials, sometimes referred to as exponential polynomials, are a family of orthogonal polynomials that are defined with regard to a certain weight function on the non-negative integers (0, 1, 2,…). There are several uses for these polynomials in the domains of mathematical analysis, probability theory, and combinatorics (Jalil, 2020).

This work investigate the use of numerical approximation techniques to address IDEs, with a particular focus on the second-kind Volterra integro-differential equations, by using shifted Legendre and Touchard polynomials as basis functions. By assuming a different-degree approximate solution utilizing shifted Legendre and Touchard polynomials, the numerical solutions of linear Volterra integro-differential equations are used in this study to examine the second-kind Volterra integro-differential equation.

1.1 General shifted Legendre polynomials

The famous Legendre polynomials are defined in interval $[0,1]$ and may be found using the following recurrence formula. The shifted Legendre polynomials are a collection of functions that are similar to the Legendre polynomials but defined on the interval $[-1,1]$ (Rahman and Islam, 2012).

$$P_{n+1}(x) = \left(\frac{2n+1}{n+1}\right).x.P_n(x) - \left(\frac{n}{n+1}\right)P_{n-1}(x); \quad n = 0,1,2,\ldots\ldots \tag{1}$$

With $P_n(x) = 1$ and $P_1(x) = x$.

For practical use of Legendre polynomials on the interval of interest $z \in [a,b]$, it is necessary to shift the defining by means of the following substitution

$$x = \frac{2x-a-b}{b-a}; \qquad a \le x \le b \tag{2}$$

The shifted Legendre polynomials in x are then obtained as follows:

$$P_0^*(x) = 1 \tag{3}$$

$$P_1^*(x) = \frac{2x-a-b}{b-a} \tag{4}$$

$$P_{n+1}^*(x) = \frac{(2n+1)(2x-a-b)}{(n+1)(b-a)}P_n^*(x) - \frac{n}{n+1}P_{n-1}^*(x); \quad n = 0,1,2 \tag{5}$$

The orthogonality conditions is:

$$\int_a^b P_n^*(x)P_m^*(x)\,dx = \begin{cases} 0, & form \neq n \\ \lambda_n form = n \end{cases} \tag{6}$$

The first few shifted Legendre polynomials are given as:

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$$\left.\begin{array}{c} P_0^* = 1 \\ \downarrow \\ P_1^* = 2x - 1 \\ \downarrow \\ P_2^*(x) = 6x^2 - 6x + 1 \\ \downarrow \\ P_3^*(x) = 20x^3 - 30x^2 + 12x - 1 \\ \downarrow \\ P_4^*(x) = 70x^4 - 140x^3 + 90x^2 - 20x + 1 \\ \downarrow \\ P_5^*(x) = 252x^5 - 630x^4 + 560x^3 - 210x^2 + 30x - 1 \\ \vdots \end{array}\right\} \quad (7)$$

1.2 Touchard polynomials

The Touchard polynomials, commonly referred to as exponential polynomials, are a binomial-type polynomial sequence that was initially researched by the French mathematician Jacques Touchard (1885–1968). They are defined as follows:

$T_n(x) = \sum_{r=0}^{n} S(n,r)p^r(x) = \sum_{r=0}^{n}\binom{n}{r}$ (8)

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, n is the degree of the polynomials, r is the index of the polynomials.

The first five Touchard polynomials are given below:

when

$$\left.\begin{array}{c} n = 0 : T_0(x) = 1 \\ n = 1 : T_1(x) = 1 + x \\ n = 2 : T_2(x) = 1 + 2x + x^2 \\ n = 3 : T_3(x) = 1 + 3x + 3x^2 + x^3 \\ n = 4 : T_4(x) = 1 + 4x + 6x^2 + 4x^3 + x^4 \\ n = 5 : T_5(x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \end{array}\right\} \quad (9)$$

and so on.

2.0 Methods of Solution

The universal integro-differential equation of this form is examined in this section:

$\varphi_{01}\xi^n(x) + \varphi_{11}\xi^{n-1}(x) + \cdots + \varphi_{n1}\xi^{n-1}(\mathrm{x}) + \gamma\int_{g(x)}^{h(x)} K(x,t)\xi(t)dt = f(x)$ (10)

subject to the conditions

$\xi = A_0, \xi'(0) = A_1, \ \xi''(0) = A_2, \ldots, \ \xi^{(n-1)}(0) = A_{n-1}$ (11)

where $K(x,t)$ and $f(x)$ are given functions, ξ'^n are unknown constants to be determined, $\varphi_{01}, \varphi_{11}, \varphi_{21}$ are real constants, and $h(x)$, $g(x)$are finite constants. The following basis functions are employed to build the usual collocation scheme for solving equations (10) and (11);

(i) Shifted Legendre polynomials and (ii) Touchard polynomials.

2.1 Standard collocation point by shifted Legendre polynomials/Method of Solution

We assume a trial solution of the form:

$\xi_N(x) = \sum_{j=0}^{N} a_j P_j^*(x)$ (12)

where N is the degree of our approximation, $P_j^*(x)(j \geq 0)$ are shifted Legendre polynomials defined in equation (7) to solve the problem given in equation (10) with the initial conditions given in equation (11), and $a_{j,}\ j = 0,1,2,\ldots,N$ are unknown constants to be determined.

Consequently, the results of differentiating equation (12) N times, yields;

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$$\left.\begin{aligned} \xi_N^{*\prime}(x) &= \sum_{j=0}^{N} a_j P_j^{*\prime}(x) \\ \xi_N^{*\prime\prime}(x) &= \sum_{j=0}^{N} a_j P_j^{*\prime\prime}(x) \\ \xi_N^{*\prime\prime\prime}(x) &= \sum_{j=0}^{N} a_j P_j^{*\prime\prime\prime}(x) \\ &\vdots \\ \xi_N^{*(n)}(x) &= \sum_{j=0}^{N} a_j P_j^{*n} \end{aligned}\right\} \quad (13)$$

The following results from substituting equations (12) and (13) into equation (10), gives;

$$\varphi_{01}\sum_{j=0}^{N} a_j P_j^{*n} + \varphi_{11}\sum_{j=0}^{N} a_j P_j^{*(n-1)} + \varphi_{21}\sum_{j=0}^{N} a_j P_j^{*(n-2)} \ldots + \varphi_{n1}\sum_{j=0}^{N} a_j\, P_j^{*}(x) + \lambda\int_{g(x)}^{h(x)} K(x,t)\sum_{j=0}^{N} a_j\, P_j^{*}(t)dt = f(x) \quad (14)$$

After the integral component of the right-hand side of Equation (14) is evaluated and simplified, the following results are obtained:

$$\varphi_{01}\sum_{j=0}^{N} a_j P_j^{*n}(x) + \varphi_{11}\sum_{j=0}^{N} a_j P_j^{*(n-1)}(x) + \varphi_{21}\sum_{j=0}^{N} a_j P_j^{*(n-2)}(x) \ldots + \varphi_{n1}\sum_{j=0}^{N} a_j\, P_j^{*}(x) + \lambda G(x) = f(x) \quad (15)$$

where $G(x) = \int_{g(x)}^{h(x)} K(x,t)\sum_{j=0}^{N} a_j\, P_j^{*}(t)dt$

Collocating equation (15) at point$x = x_k$, hence:

$$\varphi_{01}\sum_{j=0}^{N} a_j P_j^{*n}(x_k) + \varphi_{11}\sum_{j=0}^{N} a_j P_j^{*(n-1)} + \varphi_{21}\sum_{j=0}^{N} a_j P_j^{*(n-2)} \ldots + \varphi_{n1}\sum_{j=0}^{N} a_j\, P_j^{*}(x) + \lambda G(x) = f(x_k) \quad (16)$$

where $x_k = \rho + \frac{(\sigma-\rho)k}{N+1}, \quad k = 1,2,\ldots,N$ (17)

Thus, equation (17) is then put into matrix form: $[A]\,[X] = [\phi]$

(18) as:

$$\begin{pmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} & \zeta_{14} & \cdots & \zeta_{1n} \\ \zeta_{21} & \zeta_{22} & \zeta_{23} & \zeta_{24} & \cdots & \zeta_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \zeta_{11}{}^{0} & \zeta_{12}{}^{0} & \zeta_{13}{}^{0}\zeta_{14}{}^{0} & \cdots & \zeta_{1n}{}^{0} & \\ \zeta_{21}{}^{0} & \zeta_{22}{}^{0} & \zeta_{11}{}^{0}\zeta_{11}{}^{0} & \cdots & \zeta_{21}{}^{0} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \zeta_{m1}{}^{n-1} & \zeta_{m2}{}^{n-1} & \zeta_{m3}{}^{n-1}\zeta_{m4}{}^{n-1} & \cdots & \zeta_{mn}{}^{n-1} & \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} f_{11} \\ f_{21} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f_{mn}{}^{n-1} \end{pmatrix} \quad (19)$$

where $f_j's$ are values of $\phi(x_k)$ and $\zeta_k{}'s$ and $\zeta_k{}^{*\prime}s$ are the coefficients of $a_j's$. The system of equations is then solved using the matrix inversion technique to determine the unknown constants.

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} & \zeta_{14} & \cdots & \zeta_{1n} \\ \zeta_{21} & \zeta_{22} & \zeta_{23} & \zeta_{24} & \cdots & \zeta_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \zeta_{11}{}^{0} & \zeta_{12}{}^{0} & \zeta_{13}{}^{0}\zeta_{14}{}^{0} & \cdots & \zeta_{1n}{}^{0} & \\ \zeta_{21}{}^{0} & \zeta_{22}{}^{0} & \zeta_{11}{}^{0}\zeta_{11}{}^{0} & \cdots & \zeta_{21}{}^{0} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \zeta_{m1}{}^{n-1} & \zeta_{m2}{}^{n-1} & \zeta_{m3}{}^{n-1}\zeta_{m4}{}^{n-1} & \cdots & \zeta_{mn}{}^{n-1} & \end{pmatrix}^{-1}\begin{pmatrix} f_{11} \\ f_{21} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f_{mn} \end{pmatrix} \quad (20)$$

The (N+1) algebraic linear equations in (N+k+1) unknown constants are thus the result of equation (17). From equation (11), two further equations are derived from the initial conditions (11); that leaves us with (N+1) unknown constants overall. These equations are then solved

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using MAPLE 18 to get the (N+1) unknown a_j (j≥0) constants, which are then reinserted into the approximate solution provided by equation (12).

2.2 Standard collocation point by Touchard polynomials

We assume an approximate solution of the following kind in order to discuss this method:

$$\xi_N(x) = \sum_{j=0}^{N} c_j\, T_j(x) \tag{21}$$

where N is the degree of our approximation, $T_j^*(x)(j \geq 0)$ are Touchard polynomials defined in equation (9) to solve the problem given in equation (10) with the initial conditions given in equation (11), and $C_{j,}\ j = 0,1,2,\ldots,N$ are unknown constants to be determined.

Consequently, the results of differentiating equation (21) N times, yields;

$$\left.\begin{aligned} \xi_N^{*\prime}(x) &= \sum_{j=0}^{N} c_j T_j^{*\prime}(x) \\ \xi_N^{*\prime\prime}(x) &= \sum_{j=0}^{N} c_j T''(x) \\ \xi_N^{*\prime\prime\prime}(x) &= \sum_{j=0}^{N} c_j T_j^{*\prime\prime\prime}(x) \\ &\vdots \\ \xi_N^{*\,(n)}(x) &= \sum_{j=0}^{N} c_j T_j^{*n} \end{aligned}\right\} \tag{22}$$

Therefore, when equation (21) and equation (22) are substituted into equation (10), the following results:

$$\varphi_{01} \sum_{j=0}^{N} c_j T_j^{*n} + \varphi_{11} \sum_{j=0}^{N} c_j T_j^{*(n-1)} + \varphi_{21} \sum_{j=0}^{N} c_j T_j^{*(n-2)} \ldots + \varphi_{n1} \sum_{j=0}^{N} c_j\, T_j^*(x) + \lambda \int_{g(x)}^{h(x)} K(x,t) \sum_{j=0}^{N} c_j\, T_j^*(t)dt = f(x) \tag{23}$$

After evaluating and simplifying the integral portion of Equation (23)'s right side, the following results are obtained:

$$\varphi_{01} \sum_{j=0}^{N} c_j T_j^{*n}(x) + \varphi_{11} \sum_{j=0}^{N} c_j T_j^{*(n-1)}(x) + \varphi_{21} \sum_{j=0}^{N} c_j T_j^{*(n-2)}(x) \ldots + \varphi_{n1} \sum_{j=0}^{N} c_j\, T_j^*(x) + \lambda G(x) = f(x) \tag{24}$$

where $G(x) = \int_{g(x)}^{h(x)} K(x,t) \sum_{j=0}^{N} c_j\, T_j^*(t)dt$

Therefore, at point$x = x_k$,, collocating equation (24), we have:

$$\varphi_{01} \sum_{j=0}^{N} c_j T_j^{*n}(x_k) + \varphi_{11} \sum_{j=0}^{N} c_j T_j^{*(n-1)} + \varphi_{21} \sum_{j=0}^{N} c_j T_j^{*(n-2)} \ldots + \varphi_{n1} \sum_{j=0}^{N} c_j\, T_j^*(x) + \lambda G(x) = f(x_k) \tag{25}$$

where $x_k = \rho + \frac{(\sigma-\rho)k}{N+1}, \quad k = 1,2,\ldots,N$ (26)

Consequently, equation (26) is subsequently converted to matrix form as;

Thus, equation (26) is then put into matrix form: $[D]\,[X] = [\phi]$ (27)

as:

$$\begin{pmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} & \zeta_{14} & \cdots & \zeta_{1n} \\ \zeta_{21} & \zeta_{22} & \zeta_{23} & \zeta_{24} & \cdots & \zeta_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \zeta_{11}{}^{0} & \zeta_{12}{}^{0} & \zeta_{13}{}^{0} & \zeta_{14}{}^{0} & \cdots & \zeta_{1n}{}^{0} \\ \zeta_{21}{}^{0} & \zeta_{22}{}^{0} & \zeta_{11}{}^{0} & \zeta_{11}{}^{0} & \cdots & \zeta_{21}{}^{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \zeta_{m1}{}^{n-1} & \zeta_{m2}{}^{n-1} & \zeta_{m3}{}^{n-1} & \zeta_{m4}{}^{n-1} & \cdots & \zeta_{mn}{}^{n-1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} f_{11} \\ f_{21} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f_{mn}{}^{n-1} \end{pmatrix} \tag{28}$$

where $f_j's$ are values of $\phi(x_k)$ and $\zeta_k{}'s$ and $\zeta_k{}^{*\prime}s$ are the coefficients of $c_j{}'s$. The system of equations is then solved using the matrix inversion technique to determine the unknown constants.

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$$\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} & \zeta_{14} & \cdots & \zeta_{1n} \\ \zeta_{21} & \zeta_{22} & \zeta_{23} & \zeta_{24} & \cdots & \zeta_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ {\zeta_{11}}^0 & {\zeta_{12}}^0 & {\zeta_{13}}^0 & {\zeta_{14}}^0 & \cdots & {\zeta_{1n}}^0 \\ {\zeta_{21}}^0 & {\zeta_{22}}^0 & {\zeta_{11}}^0 & {\zeta_{11}}^0 & \cdots & {\zeta_{21}}^0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ {\zeta_{m1}}^{n-1} & {\zeta_{m2}}^{n-1} & {\zeta_{m3}}^{n-1} & {\zeta_{m4}}^{n-1} & \cdots & {\zeta_{mn}}^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} f_{11} \\ f_{21} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f_{mn} \end{pmatrix} \quad (29)$$

Similarly, the (N+1) algebraic linear equations in (N+k+1) unknown constants are thus the result of equation (26). From equation (11), two further equations are derived from the initial conditions (11). That leaves us with (N+1) unknown constants overall. These equations are then solved using MAPLE 18 to get the (N+1) unknown c_j (j≥0) constants, which are then reinserted into the approximate solution provided by equation (21).

3.0 Numerical; Experiments

3.1 Numerical Example 1

Examine the following second-kind Volterra integro-differential equation (Wazwaz, 2015):

$$\xi''(x) = x\cos x - 2\sin x + \int_0^x t\xi(t)dt \quad (30)$$

subject to conditions:

$$\xi(0) = 0, \xi'(0) = 1 \quad (31)$$

The exact solution is:

$$\xi(x) = \sin x \quad (32)$$

3.1.1 Standard collocation point by shifted Legendre polynomials for case N = 4

In this case, we anticipated a trial solution that looks like this:

$\xi_N(x) = \sum_{j=0}^{N} a_j P_j^*(x)$ (33)

Following further simplification, the approximate result is as follows:

$$\xi_4(x) = 2.4710^{-11} + 0.9999999999x + 0.0038064440x^2 - 0.1783764209x^3 + 0.0161883673x^4 \quad (34)$$

3.1.2 Standard collocation point by Touchard polynomials for case N = 4

Here, we assume a trial solution of the form:

$\xi_N(x) = \sum_{j=0}^{N} c_j T_j(x)$ (35)

Consequently, we derive the approximate solution after simplification:

$$\xi_4(x) = 1.000000010 + 1.005399707x + 0.5054264910x^2 + 0.1508539402x^3 + 0.062366837x^4 \quad (36)$$

3.1.3 Standard collocation point by shifted Legendre polynomials for case N = 5 in case of Example 1 in 3.1

The same steps and procedures involved for case N = 4 in terms of obtaining its unknown constants, also the same routine procedures and steps are followed for case N = 5,

And approximate solution is given as

$$\xi_5(x) = 0.9999999996 + 1.000610678x + 0.4994590700x^2 + 0.1689177624x^3 + 0.03714397512x^4 + 0.01268476423x^5 \quad (37)$$

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3.1.4 Standard collocation point by Touchard polynomials for case N = 5 in case of Example 1 in 3.1

Similarly, the same steps and procedures involved for case N = 4 in terms of obtaining its unknown constants, also the same routine procedures and steps are followed for case N = 5,

And approximate solution is given as;

$$\xi_5(x) = 1.000000083 + 1.000610685x + 0.4994589500x^2 + 0.1689176992x^3 + 0.037143985x^4 + 0.012684763x^5 \quad (38)$$

Table 3.1: Result of the exact, shifted Legendre polynomial and Touchard polynomial of Example 1 when $N = 4$

X	Exact	Shifted Legendre polynomial	Touchard polynomial
0.0	0.0000000000	-0.0000000002	-0.0000000001
0.1	0.09983341665	0.09986130659	0.09986130682
0.2	0.1986693308	0.1987511475	0.1987511477
0.3	0.2955202067	0.2956575420	0.2956575424
0.4	0.3894183423	0.3896073621	0.3896073623
0.5	0.4794255386	0.4796663311	0.4796663315
0.6	0.5646424734	0.5649390251	0.5649390256
0.7	0.6442176872	0.6445688720	0.6445688727
0.8	0.7173560909	0.7177381517	0.7177381527
0.9	0.7833269096	0.7836679967	0.7836679978
1.0	0.8414709848	0.8416183906	0.8416183919

Table 3.2: Error in shifted Legendre polynomial and Touchard polynomial of Example 1 when $N = 4$

X	Error of Shifted Legendre polynomial	Error of Touchard polynomial
0.0	0.0000000002	0.0000000001
0.1	0.00002788994	0.00002789017
0.2	0.0000818167	0.0000818169
0.3	0.0001373353	0.0001373357
0.4	0.0001890198	0.0001890200
0.5	0.0002407925	0.0002407929
0.6	0.0002965517	0.0002965522
0.7	0.0003511848	0.0003511855
0.8	0.0003820608	0.0003820618
0.9	0.0003410871	0.0003410882
1.0	0.0001474058	0.0001474071

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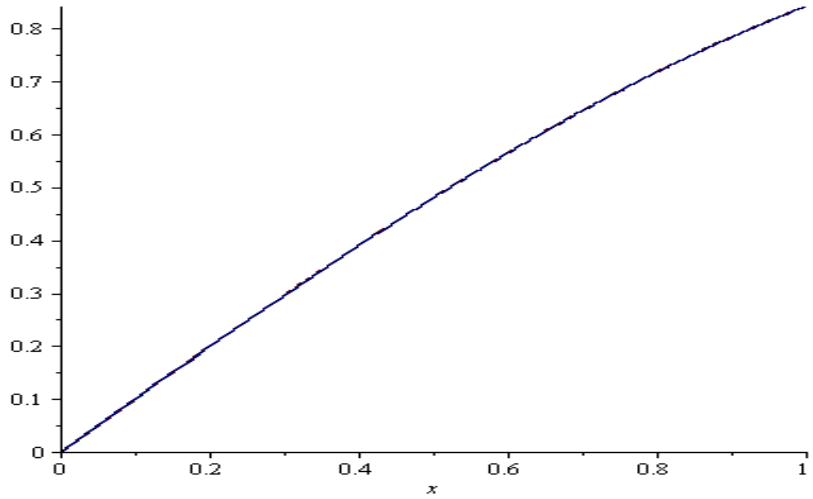

Figure 3.1: Graph of the comparison of shifted Legendre polynomial and exact solutions for Example 1 when $N = 4$

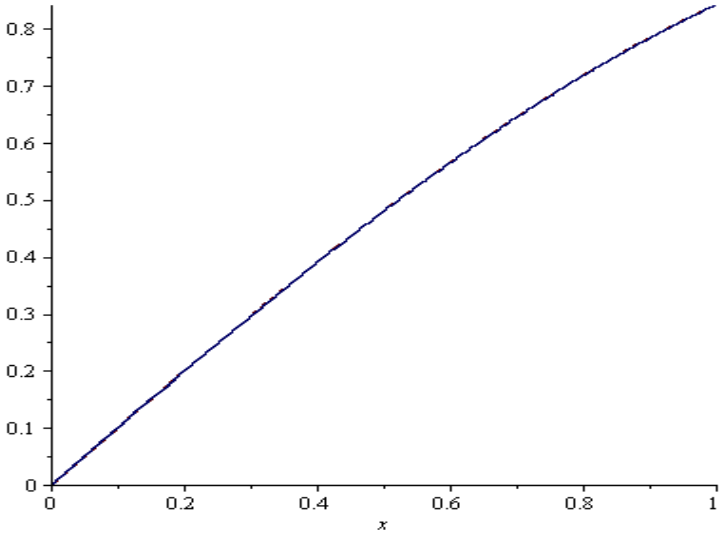

Figure 3.2: Graph of the comparison of Touchard polynomial and exact solutions for Example 1 when $N = 4$

Table 3.3: Result of the exact, shifted Legendre polynomial and Touchard polynomial of Example 1 when $N = 5$

X	Exact	Shifted Legendre polynomial	Touchard polynomial
0.0	0.000000000	-0.0000000004	0.000000000
0.1	0.09983341665	0.09983563755	0.09983563785
0.2	0.1986693308	0.1986757668	0.1986757670
0.3	0.2955202067	0.2955309826	0.2955309825
0.4	0.3894183423	0.3894332297	0.3894332296
0.5	0.4794255386	0.4794445357	0.4794445352
0.6	0.5646424734	0.5646657429	0.5646657421
0.7	0.6442176872	0.6442452420	0.6442452412
0.8	0.7173560909	0.7173877056	0.7173877042
0.9	0.7833269096	0.7833628190	0.7833628177
1.0	0.8414709848	0.8415140160	0.8415140149

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Table 3.4: Error in shifted Legendre polynomial and Touchard polynomial of Example 1 when $N = 5$

X	Error of Shifted Legendre polynomial	Error of Touchard polynomial
0.0	0.0000000004	0.000000000
0.1	0.00000222090	0.00000222120
0.2	0.0000064360	0.0000064362
0.3	0.0000107759	0.0000107758
0.4	0.0000148874	0.0000148873
0.5	0.0000189971	0.0000189966
0.6	0.0000232695	0.0000232687
0.7	0.0000275548	0.0000275540
0.8	0.0000316147	0.0000316133
0.9	0.0000359094	0.0000359081
1.0	0.0000430312	0.0000430301

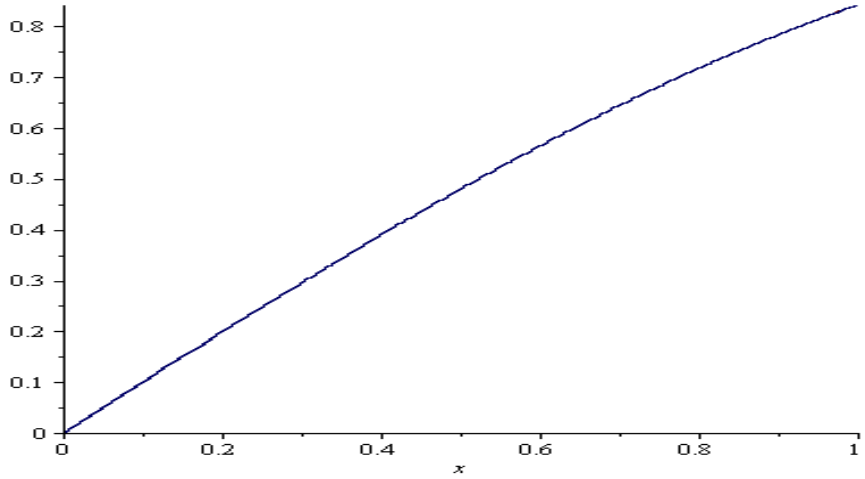

Figure 3.3: Graph of the comparison of shifted Legendre polynomial and exact solutions for Example 1 when $N = 5$

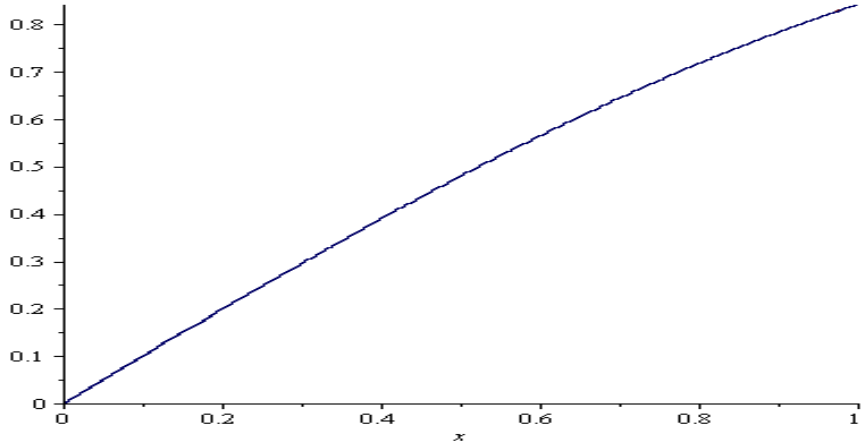

Figure 3.4: Graph of the comparison of Touchard polynomial and exact solutions for Example 1 when $N = 5$

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3.2 Numerical Example 2

In Aggarwai *et al.* (2019), the linear integro-differential equation is examined:

$$\xi'''(x) = 1 + x - 2x^2 + \int_0^x (x-t)\,\xi(x)\,dt \tag{39}$$

subject to the condition:

$$\xi(0) = 5,\ \xi'(0) = 1,\ \xi''(0) = 1; \tag{40}$$

whose exact equation is: $\xi(x) = 4 + e^x$. (41)

3.2.1 Employing the shifted Legendre polynomial as the basis function.

Let us consider the shifted Legendre polynomial as the assumed solution, which is given as

$$\xi(x) = \sum_{k=0}^{N} c_k P_k^* \tag{42}$$

$$\begin{aligned}\xi(\mathrm{x}) &= 4.804172734 + 1.687717390x + 0.2085810459(2x-1)^2 \\ &+0.0135585994(2x-1)^3 + 0.02033789910((2x-1)^2-1)(2x-1) \\ &+0.000804819983(2x-1)^4 + 0.002414459949((2x-1)^2-1)(2x-1)^2 \\ &+0.0003018074936((2x-1)^2-1)^2\end{aligned} \tag{43}$$

When $N = 5$,

$$\begin{aligned}\xi(x) &= 4.803112403 + 1.690639889x + 0.2098730606(2\mathrm{x}\text{-}1)^2 \\ &+0.01391899113(2x-1)^3 + 0.02087848670\left((2x-1)^2-1\right) \\ &(2x-1) + 0.00098302575(2x-1)^4 + 0.00294907725((2\mathrm{x}\text{-}1)^2-1)(2x-1)^2 \\ &+0.0003686346562((2x-1)^2-1)^2 + 0.000049499170(2x-1)^5 + 0.000247495850 \\ &((2\mathrm{x}\text{-}1)^2-1)(2x-1)^3 + 0.00009281094375((2x-1)^2-1)^2(2x-1)\end{aligned} \tag{44}$$

3.2.2 Touchard Polynomial as the Basis Function

Consider the Touchard polynomial, the assumed solution, which is provided as:

$$\xi(x) = \sum_{k=0}^{N} c_k T_k^* \tag{45}$$

$$\xi(x) = 5 + x + 0.4999999999x^2 + 0.1584971892x^3 + 0.05633740008x^4 \tag{46}$$

When $N = 5$,

$$\xi(x) = 5.000000005 + x + 0.5000000000x^2 + 0.1684757591x^3 + 0.03762731660x^4 + 0.01247379400x^5 \tag{47}$$

Table 3.5: Results of the exact solution, shifted Legendre polynomial and Touchard polynomial of Example 2 when $N = 4$

X	Exact	Shifted Legendre polynomial	Touchard polynomial
0.0	5.00000000	5.000000001	5.00000000
0.1	5.105170918	5.105164130	5.105164131
0.2	5.221402758	5.221358118	5.221358118
0.3	5.349858808	5.349735757	5.349735757
0.4	5.491824698	5.491586059	5.491586057
0.5	5.648721271	5.648333236	5.648333237
0.6	5.822118800	5.821536721	5.821536720
0.7	6.013752707	6.012891145	6.012891146

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0.8	6.225540928	6.224226360	6.224226360
0.9	6.459603111	6.457507418	6.457507419
1.0	6.718281828	6.714834589	6.714834589

Table 3.6: Error in shifted Legendre polynomial and Touchard polynomial of Example 2 when $N = 5$

X	Error of Shifted Legendre polynomial	Error of Touchard polynomial
0.0	0.000000001	0.00000000
0.1	0.000006788	0.000006787
0.2	0.000044640	0.000044640
0.3	0.000123051	0.000123051
0.4	0.000238639	0.000238641
0.5	0.000388035	0.000388034
0.6	0.000582079	0.000582080
0.7	0.000861562	0.000861561
0.8	0.001314568	0.001314568
0.9	0.002095693	0.002095692
1.0	0.003447239	0.003447239

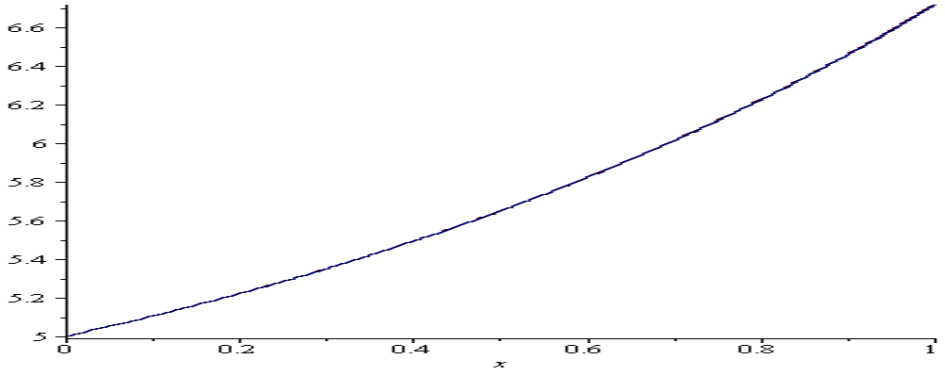

Figure 3.5: Graph comparing the shifted Legendre polynomial and precise for $N = 5$ for Example 2

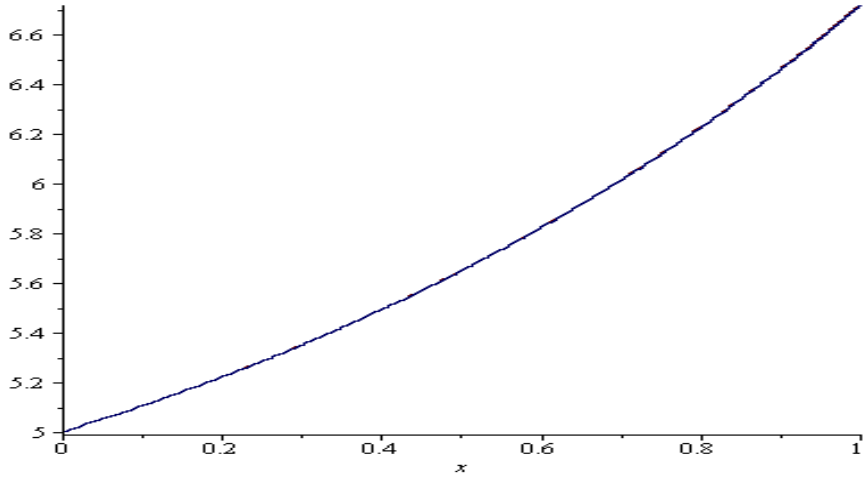

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Figure 3.6: Graph comparing the Touchard polynomial and exact solution when $N = 5$ for Example 2

Table 3.7: Result of the exact solution, shifted Legendre polynomial and Touchard polynomial of Example 2 when $N = 5$

X	Exact	Shifted Legendre polynomial	Touchard polynomial
0.0	5.00000000	5.00000000	5.000000005
0.1	5.105170918	5.105172362	5.105172369
0.2	5.221402758	5.221412002	5.221412007
0.3	5.349858808	5.349883938	5.349883942
0.4	5.491824698	5.491873438	5.491873445
0.5	5.648721271	5.648800983	5.648800988
0.6	5.822118800	5.822237225	5.822237231
0.7	6.013752707	6.013917973	6.013917980
0.8	6.225540928	6.225759149	6.225759156
0.9	6.459603111	6.459871761	6.459871766
1.0	6.718281828	6.718576869	6.718576875

Table 3.8: Error in shifted Legendre polynomial and Touchard polynomial of Example 2 when $N = 5$

X	Error of Shifted Legendre polynomial	Error of Touchard polynomial
0.0	0.000000000	0.000000005
0.1	0.000001444	0.000001451
0.2	0.000009244	0.000009249
0.3	0.000025130	0.000025134
0.4	0.000048740	0.000048747
0.5	0.000079712	0.000079717
0.6	0.000118425	0.000118431
0.7	0.000165266	0.000165273
0.8	0.000218221	0.000218228
0.9	0.000268650	0.000268655
1.0	0.000295041	0.000295047

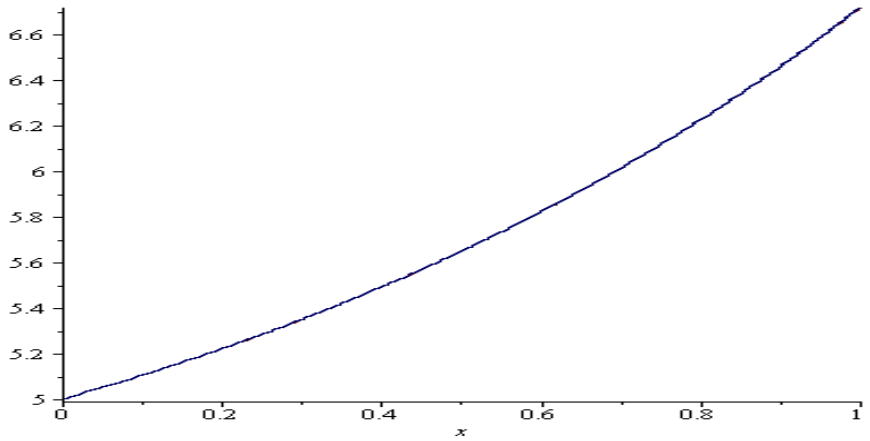

Figure 3.7: Graph of the comparison of shifted Legendre polynomial and exact for Example 2 when $N = 5$

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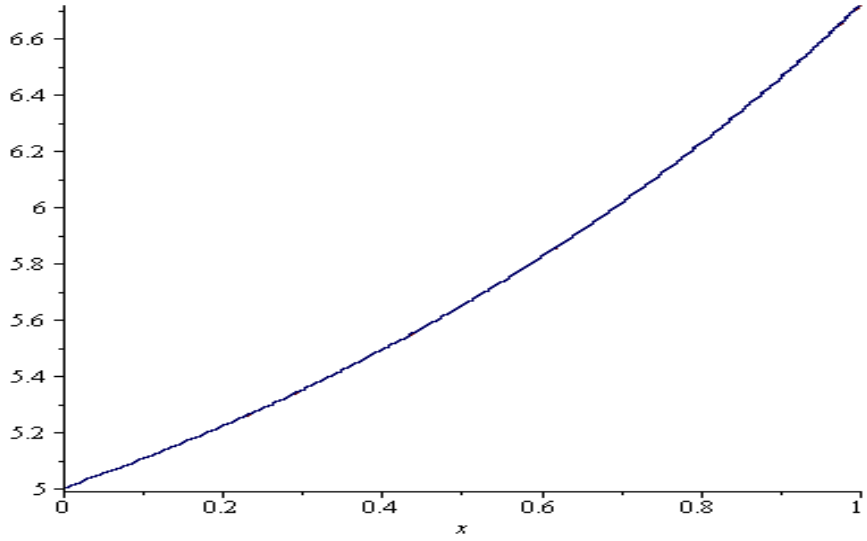

Figure 3.8: Graph of the comparison of Touchard polynomial and exact solution for Example 2 when $N = 5$

3.3 Numerical Example 3

Considering the second-kind linear Volterra integro-differential equation, as stated by Aggarwai *et al*. (2020):

$$\xi''(x) = 1 + x + \int_0^x (x-t)\xi(t)dt\,. \tag{48}$$

subject to conditions

$$\xi(0) = 1,\ \xi'(0) = 1\,. \tag{49}$$

The exact solution

$$\xi(x) = e^x\,. \tag{50}$$

3.3.1 The basis function is the shifted Legendre polynomial.

Let us consider the shifted Legendre polynomial as the assumed solution, which is given as:

$$\xi(x) = \sum_{k=0}^{N} c_k P_k^* \tag{51}$$

$$\begin{aligned}\xi(\mathrm{x}) &= 0.8033425906 + 1.690784082x + .2095292228(2x-1)^2 \\ &+0.1376292972\text{e-}1(2x-1)^3 + 0.2064439458\text{e-}1((2x-1)^2-1)(2x-1) \\ &+0.89111739\text{e-}3(2x-1)^4 + 0.267335217\times10^{-2}((2x-1)^2-1)(2x01)^2 \\ &+0.3341690212\times10^{-3}((2x-1)^2-1)^2\end{aligned}\,. \tag{52}$$

When $N = 5$,

$$\begin{aligned}\xi(x) &= 0.8032032448 + 1.690148255x + .2097928666(2\text{x-}1)^2 \\ &+0.1393086357\times10^{-1}(2x-1)^3 + 0.2089629536\times10^{-1}\left((2x-1)^2-1\right) \\ &(2x-1) + 0.98954123\times10^{-3}(2x-1)^4 + 0.00296862369((2\text{x-}1)^2-1)(2x-1)^2 \\ &+0.3710779612\text{e-}3((2x-1)^2-1)^2 + 0.00005478910(2x-1)^5 + 0.00027394550 \\ &((2\text{x-}1)^2-1)(2x-1)^3 + 0.0001027295625((2x-1)^2-1)^2(2x-1)\end{aligned}\,. \tag{53}$$

3.3.2 Touchard polynomial as the basis function

Consider the Touchard polynomial, the assumed solution, which is provided as:

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$$\xi(x) = \sum_{k=0}^{N} c_k T_k^* \tag{54}$$

$$\xi(x) = 0.9999999998 + 1.000000000x + 0.5054295654x^2 + 0.1505021598x^3 + 0.0623782170x^4 \tag{55}$$

When $N = 5$,

$$\xi(x) = 1 + x + 0.4987985587x^2 + 0.1707633992x^3 + 0.0347507448x^4 + 0.0138068572x^5 \tag{56}$$

Table 3.9: Result of the exact solution, shifted Legendre polynomial and Touchard polynomial of Example 3 when $N = 5$

X	Exact	Shifted Legendre polynomial	Touchard polynomial
0.0	1.00000000	1.000000001	0.9999999998
0.1	1.105170918	1.105211037	1.105211036
0.2	1.221402758	1.221521007	1.221521005
0.3	1.349858808	1.350057484	1.350057483
0.4	1.491824698	1.492097752	1.492097750
0.5	1.648721271	1.649068801	1.649068800
0.6	1.822118800	1.822547327	1.822547328
0.7	2.013752707	2.014259739	2.014259738
0.8	2.225540928	2.226082147	2.226082146
0.9	2.459603111	2.460040372	2.460040370
1.0	2.718281828	2.718309943	2.718309942

Table 3.10: Error in Example 3's shifted Legendre and Touchard polynomials when $N = 4$.

X	Error of Shifted Legendre polynomial	Error of Touchard polynomial
0.0	0.000000001	0.0000000002
0.1	0.000040119	0.000040118
0.2	0.000118249	0.000118247
0.3	0.000198676	0.000198675
0.4	0.000273054	0.000273052
0.5	0.000347530	0.000347529
0.6	0.000428527	0.000428528
0.7	0.000507032	0.000507031
0.8	0.000541219	0.000541218
0.9	0.000437261	0.000437259
1.0	0.000028115	0.000028114

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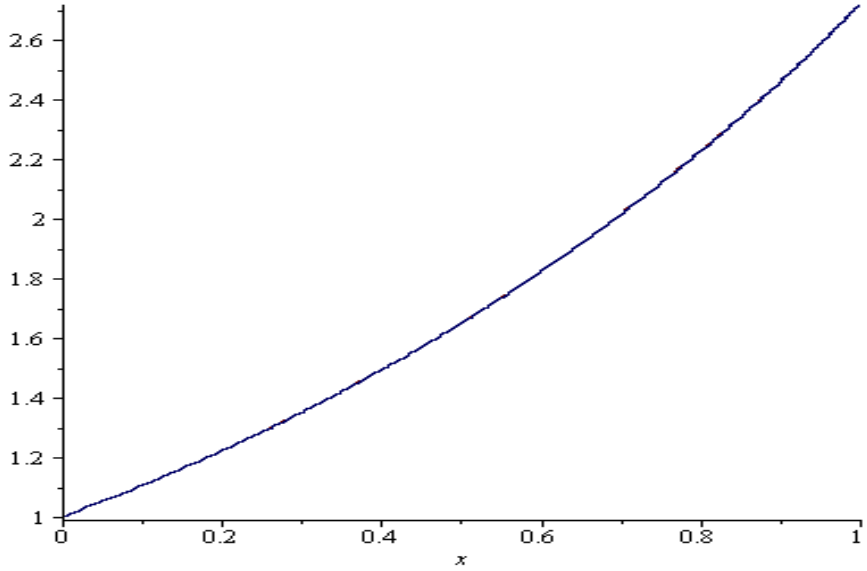

Figure 3.9: Graph comparing the shifted Legendre polynomial and exact solution for Example 3 when $N = 4$.

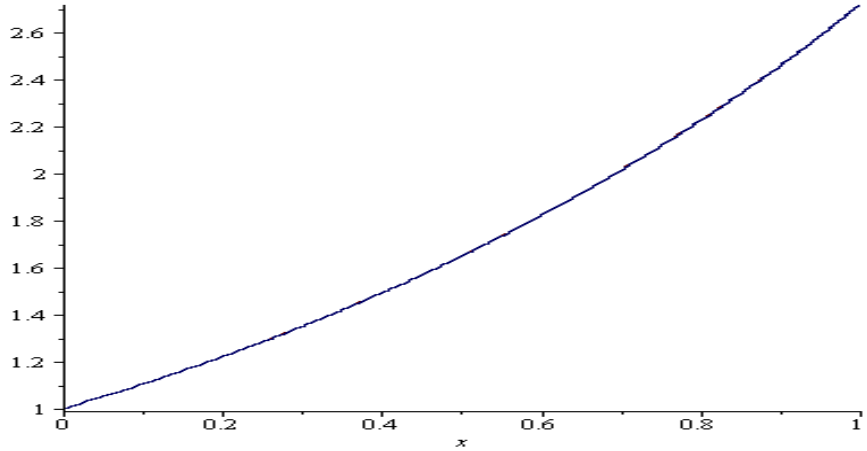

Figure 3.10: Graph showing the Touchard polynomial and exact solution comparison for Example 3 when $N = 4$.

Table 3.11: Result of the exact solution, shifted Legendre polynomial and Touchard polynomial of Example 3 when $N = 5$

X	Exact	Shifted Legendre polynomial	Touchard polynomial
0.0	1.00000000	0.9999999999	1.000000000
0.1	1.105170918	1.105162363	1.105162362
0.2	1.221402758	1.221378069	1.221378068
0.3	1.349858808	1.349817514	1.349817514
0.4	1.491824698	1.491767628	1.491767628
0.5	1.648721271	1.648648450	1.648648451
0.6	1.822118800	1.822029693	1.822029693
0.7	2.013752707	2.013647312	2.013647312
0.8	2.225540928	2.225420076	2.225420074

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0.9	2.459603111	2.459466126	2.459466125
1.0	2.718281828	2.718119561	2.718119560

Table 3.12: Error in shifted Legendre polynomial and Touchard polynomial of Example 3 when $N = 5$

X	Error of Shifted Legendre polynomial	Error of Touchard polynomial
0.0	0.0000000001	0.00000000
0.1	0.000008555	0.000008556
0.2	0.000024689	0.000024690
0.3	0.000041294	0.000041294
0.4	0.000057070	0.000057070
0.5	0.000072821	0.000072820
0.6	0.000089107	0.000089107
0.7	0.000105395	0.000105395
0.8	0.000120852	0.000120854
0.9	0.000136985	0.000136986
1.0	0.000162267	0.000162268

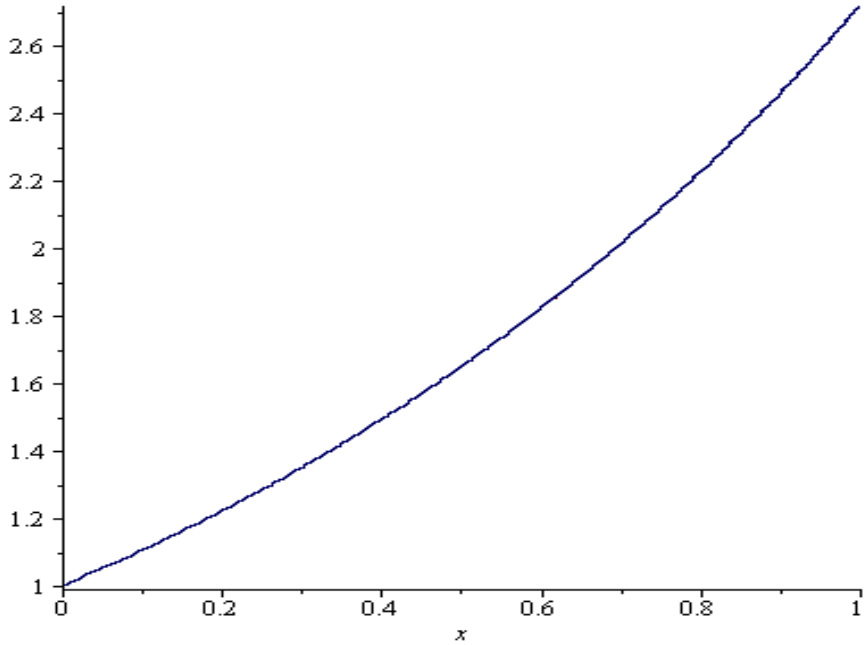

Figure 3.11: Graph of the comparison of shifted Legendre polynomial and exact solution for Example 3 when $N = 5$.

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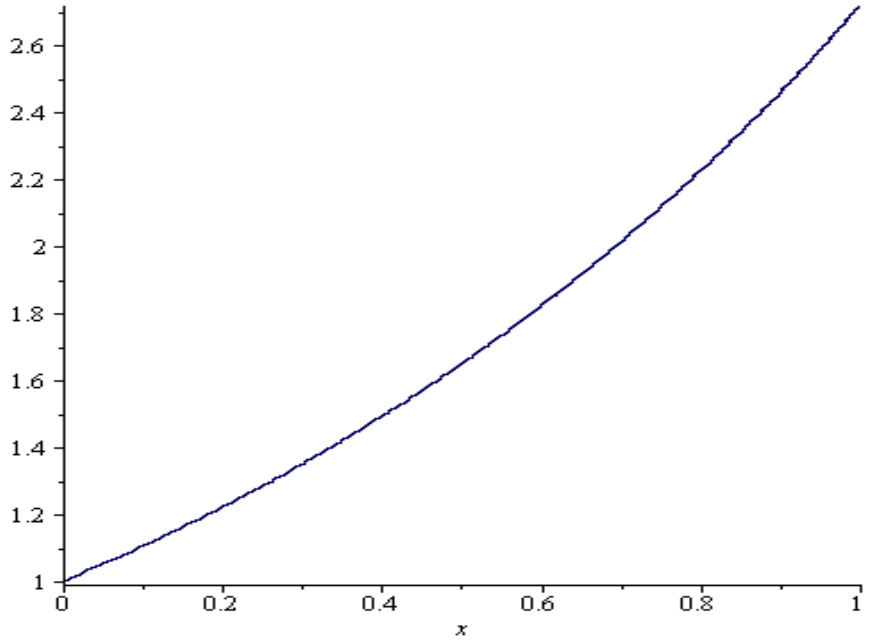

Figure 3.12: Graph of the comparison of Touchard polynomial and exact solution for Example 3 when $N = 5$

4.0 Conclusion

The Volterra integro-differential equations have been numerically solved by utilizing shifted Legendre and Touchard polynomials as trial functions. After that, the specified beginning or boundary conditions are subjected to the basis or trial functions. This study is concluded by establishing a new numerical method for the approximate solution of Volterra linear integro-differential equations using MAPLE 18 software. The study examined three numerical tests to verify the method's accuracy and effectiveness, and the outcomes are shown in tables and graphs. Due to its faster convergence to the exact solution, the Touchard polynomial was shown to be a more accurate and efficient basis function than the shifted Legendre polynomial for the solution of Volterra integro-differential equations. The method has proven to be quite successful in all the cases because it quickly converges to the precise solution. Hence, the new numerical scheme has been validated by the results of our effort thus far. Therefore, it is highly recommended to solve the linear integro-differential equations of Volterra using Touchard polynomial or shifted Legendre polynomial as the numerical approach method considered in this paper.

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