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ON THE DEGREE SEQUENCE AND SIZE OF RELATIVE PRIME COPRIME GRAPHS OF GROUP Z_{2p}

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ABSTRACT

This research conducts a comprehensive structural analysis of the relative prime coprime graph constructed on the cyclic group Z_{2p}, where p is an odd prime. The study successfully establishes precise combinatorial formulas for the graph's basic properties, revealing a clear separation in its connectivity: vertices are partitioned into two distinct classes based on their orders, with one class acting as universal vertices of degree $2p-1$ and the other having a significantly lower degree of p. This exact characterization of the vertex degrees directly enables the derivation of a closed-form expression for the graph's size, proving the total number of edges is $\frac{p(3p-1)}{2}$. Furthermore, the graph is conclusively shown to be non-complete, as vertices of order 2p are not adjacent to each other.

Keywords: graph theory, completeness, Coprime Graph, Relative Coprime Graph, vertex degree.

1.0 INTRODUCTION

The foundational concept of the coprime graph for a group was established by Dorbidi (2016), who defined it as a graph whose vertices are the elements of a group, with two distinct vertices adjacent if and only if their orders are coprime. This early work focused on classifying groups whose coprime graphs are complete partite or planar, and began the study of their automorphism groups. Building on this, Abd Rhani $et\ al.$ (2017) introduced a key variant: the relative coprime graph associated with a subset H of a group. In this graph, vertices are connected if their orders are coprime and at least one of them belongs to H. Their research identified fundamental graph-
theoretic properties such as the dominating number, the degree of a dominating set, and the independence number, and made observations on the graph's regularity.

The period from 2019 to 2021 saw a significant expansion of research into specific families of finite groups. Banerjee (2019) conducted a comprehensive investigation of the prime coprime graph, denoted $\Theta(G)$, establishing a direct link between its properties and t hose of the underlying group G. This work provided a necessary and sufficient condition for $\Theta(G)$ to be Eulerian for any finite group G. It also extensively analysed $\Theta(G)$ for cyclic groups Z_n and dihedral groups D_n, determining conditions for these graphs to be connected, complete, planar, and Hamiltonian. Furthermore, this study explored the vertex connectivity of $\Theta(Z_n)$ and computed the signless Laplacian spectrum for groups of order pq or p^m, where p and q are distinct primes. This line of inquiry was continued and refined by Banerjee and Adhikari (2021), who further solidified the relationship between Θ(G) and G. Their work reiterated and expanded upo

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n the analysis of Eulerian conditions, conne ctivity, and spectral properties for Z_n and D_n, providing a robust framework for understan ding these structures. Concurrently, Sehgal and Singh (2020) derived precise formulas f or the degree of a vertex in the coprime ord er graph of finite Abelian and dihedral group s. They also advanced the spectral analysis of these graphs by investigating the Laplaci an spectrum for finite Abelian p-groups, $Z_{p^t} \times Z_{q^s}$, and dihedral groups D_{p^n} .

Recent research has delved deeper into spe cific graph parameters and broader generali zations. Hamm and Way (2021) calculated t he independence number of the coprime gra ph for dihedral groups and provided a chara cterization of the groups for which the copri me graph is perfect. In a significant contribut ion, Hao *et al*. (2022) characterized finite gr oups whose coprime order graphs are comp lete and classified those whose graphs are planar, generalizing some of Banerjee's earli er results. They also computed the vertex-connectivity for the coprime order graphs of cyclic, dihedral, and generalized quaternion groups, thereby answering an open questio n from prior research. A very recent and nov el direction in this field involves the applicati on of topological indices. Syarifudin *et al*. (2 023) investigated general formulas for vario us topological indices such as the Zagreb, Wiener, and Harary indices for the relative c oprime graph of dihedral groups, linking dist ance and vertex degree measurements to th e graph's structural properties. Most recently , Romdhini *et al*. (2025) have introduced the concept of reverse topological indices for re latively prime coprime graphs. Their work de fines and describes properties of reverse Ha rmonic, reverse Randić, and reverse SK indi ces, based on the order of elements within t he integers modulo group. Despite this exte nsive body of work, a detailed analysis of th e degree sequence and the exact size of the relative prime coprime graph for the specific family of cyclic groups Z_{2p} remained unexpl ored, presenting a clear gap in the literature. This paper aims to fill this gap by providing a complete characterization of the relative pr ime coprime graph for Z_{2p}. We establish a fu ndamental partition of the group elements b ased on their orders, which serves as the co rnerstone for our analysis. The remainder of this article is structured as follows. Section 2 provides the necessary definitions and pre liminary concepts. In Section 3, we present our main results: deriving explicit formulas f or the degree of every vertex, calculating th e total number of edges (the size) of the gra ph, and proving that the graph is not comple te. The paper concludes with a discussion of the implications of our findings and potentia l directions for future research.

2. PRELIMINARIES

Definition 2.1 (Cyclic Group; Gallian, 2021). A cyclic group is a group that can be generated by a single element. That is, there exists an element g in G such that every element in G is a power of g (or a multiple, if the operation is addition). Such a group is denoted ⟨g⟩.

Definition 2.2 (Graph; West, 2001). A graph $\Gamma = (V, E)$ is an ordered pair consisting of a set V of vertices and a set E of edges, which are 2-element subsets of V.

Definition 2.3 (Vertex Degree; Bondy & Murty, 2008). The degree of a vertex v in a graph, denoted deg(v), is the number of edges incident to v.

Definition 2.4 (Complete Graph; Diestel, 2017). A complete graph is a simple graph in which every pair of distinct vertices is connected by a unique edge. A complete graph on n vertices is denoted K_n .

Definition 2.5 (Coprime Graph of a Group; Dorbidi, 2016). The coprime graph of a group G is defined as a graph whose vertices are elements of G, and two distinct vertices x and y are adjacent if and only if $gcd(o(x), o(y)) = 1$.

Definition 2.6 (Relative Prime Coprime Graph; Abd Rhani *et al*., 2017). Let Z_n be the integers modulo n and let H be a subgroup of Z_n. The relative prime coprime

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graph of Z_n with respect to H, denoted $\Gamma_{\{Z_n,H\}}$, is defined as follows:

- The vertex set is the elements of Z_n.
- Two distinct vertices $x \neq y$ are adjacent if and only if $gcd(|x|,|y|) = 1$ or $gcd(|x|,|y|) = p$, and either $x \in H$ or $y \in H$, where $|x|$ denotes the order of the element x.

In this paper, we focus on the case where $H = Z_{2p}$, which simplifies the adjacency condition.

3. MAIN RESULTS

The following lemmas establish a fundamental partition of the group Z_{2p} based on element orders, which is crucial for analyzing the graph $\Gamma_{\{Z_n,H\}}$.

Lemma 3.1 (Order of Elements in Z_{2p}). Let $n = 2p$, where $p > 2$ is a prime number. The elements of Z_{2p} can be partitioned into four distinct subsets:

$P_1 = \{0\}$, $P_2 = \{p\}$, $P_3 = \{x \text{ in } Z_{2p} \mid x \text{ is even, } x \text{ not in } \{0,p\}\}$, $P_4 = \{x \in Z_{2p} \mid x \text{ is odd}, x \neq p\}$

For any x in Z_{2p}, the order $o(x)$ is given by: $o(x) = \begin{cases} 1, & \text{if } x \in P_1 \\ 2, & \text{if } x \in P_2 \\ p, & \text{if } x \in P_3 \\ 2p, & \text{if } x \in P_4 \end{cases}$

Proof

Let x be in Z_{2p}. The order is given by $o(x) = \frac{2p}{gcd(x,2p)}$. We proceed by the following cases:

Case 1: $x \in P_1$. Then $x = 0$, the identity element. Thus, $o(0) = 1$.

Case 2: $x \in P_2$. Then $x = p$. Since $gcd(p,2p) = p$, we have $o(p) = \frac{2p}{p} = 2$.

Case 3: $x \in P_3$. Then x is even and x not in $\{0,p\}$, so $x = 2k$ for $1 \leq k < p$. Since p is prime and $k < p$, $gcd(k,p) = 1$. Thus, $gcd(2k,2p) = 2$, $o(x) = \frac{2p}{2} = p$.

Case 4: $x \in P_4$. Then x is odd and $x \neq p$. Thus, $gcd(x,2p) = 1$, $o(x) = \frac{2p}{1} = 2p$.

In all cases, the value of $o(x)$ matches the stated classification.

Lemma 3.2 (Partition Properties). Let Z_{2p} be the cyclic group of order $2p$ where $p > 2$ is prime. The subsets P_1, P_2, P_3, P_4 defined in Lemma 3.1 form a partition of Z_{2p} with the following cardinalities: $|P_1| = 1, |P_2| = 1, |P_3| = p - 1$, and $|P_4| = p - 1$.

Proof

Z_{2p} has $2p$ elements. There is 1 element of order 1 (P_1) and 1 element of order 2 (P_2). The number of elements of order p is $\varphi(p) = p - 1$ (P_3), and the number of elements of order $2p$ is $\varphi(2p) = \varphi(2)\varphi(p) = 1 \cdot (p - 1) = p - 1$ (P_4). The sum $1 + 1 + (p - 1) + (p - 1) = 2p$ confirms the partition is complete.

We now analyse the graph $\Gamma_{\{Z_n,H\}} = \Gamma_{\{Z_n,Z_{2p}\}}$, where the subgroup condition is trivial since H is the entire group. The adjacency rule simplifies to: two distinct vertices x and y are adjacent if and only if $gcd(o(x),o(y)) = 1$ or $gcd(o(x),o(y)) = p$.

Theorem 3.3 (Vertex Degrees). Let $p > 2$ be an odd prime and let $\Gamma_{\{Z_n,H\}}$ be the relative prime coprime graph of Z_{2p} (with $H = Z_{2p}$). With the partition P_1, P_2, P_3, P_4 from Lemma 3.1, the degree of a vertex $v \in \Gamma_{\{Z_n,H\}}$ is given by: $\deg(v) = \begin{cases} 2p - 1, & \text{if } v \in P_1 \cup P_3 \\ p, & \text{if } v \in P_2 \cup P_4 \end{cases}$

Proof

We determine the neighbours of a vertex v by considering its order.

Case 1: $v \in P_1$ $(o(v) = 1)$. For any $y \neq v$, $gcd(1,o(y)) = 1$. Thus, v is adjacent to every other vertex. Hence, $deg(v) = 2p - 1$.

Case 2: $v \in P_2$ $(o(v) = 2)$. Let $y \neq v$, then we have the following in the table:

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$o(y)$	gcd $(o(v), o(y))$	v is djacent to y
1	$(2,1) = 1$	yes
2	$(2,2) = 2$	no
p	$(2,p) = 1$	yes
$2p$	$(2,2p) = 2$	no

Thus, neighbors are $P_1 \cup P_3$, so $deg(v) = |P_1| + |P_3| = 1 + (p-1) = p$.

Case 3: $v \in P_3$ $(o(v) = p)$. Let $y \neq v$, then we have the following in the table:

$o(y)$	gcd $(o(v), o(y))$	v is djacent to y
1	$(p,1) = 1$	yes
2	$(p,2) = 1$	yes
p	$(p,p) = p$	yes
$2p$	$(p,2p) = p$	yes

Thus, v is adjacent to every other vertex, so $deg(v) = 2p - 1$.

Case 4: $v \in P_4$ $(o(v) = 2p)$. Let $y \neq v$, then we have the following in the table:

$o(y)$	gcd $(o(v), o(y))$	v is djacent to y
1	$(2p,1) = 1$	yes
2	$(2p,2) = 2$	no
p	$(2p,p) = p$	yes
$2p$	$(2p,2p) = p$	no

Thus, neighbors are $P_1 \cup P_3$, so $deg(v) = |P_1| + |P_3| = 1 + (p-1) = p$.

Example 3.1. Let $p = 5$, so $Z_{10} = \{0,1,2,\ldots,9\}$.

$P_1 = \{0\}, P_2 = \{5\}, P_3 = \{2,4,6,8\}, P_4 = \{1,3,7,9\}$

By Theorem 3.3: $deg(0) = deg(2) = deg(4) = deg(6) = deg(8) = 2 \cdot 5 - 1 = 9$, $deg(5) = deg(1) = deg(3) = deg(7) = deg(9) = 5$

The relative prime coprime graph for Z_{10}. Vertices in P_1 and P_3 have degree 9; vertices in P_2 and P_4 have degree 5.

Theorem 3.4 (Size of $\Gamma_{\{Z_n,H\}}$). Let p > 2 be an odd prime and let $\Gamma_{\{Z_n,H\}}$ be the relative prime coprime graph of Z_{2p}. The number of edges in $\Gamma_{\{Z_n,H\}}$ is $\left|E\left(\Gamma_{\{Z_n,H\}}\right)\right| = \frac{p(3p-1)}{2}$

Proof

From Theorem 3.3: Vertices in $P_1 \cup P_3$ have degree $2p - 1$. There are $|P_1| + |P_3| = 1 + (p-1) = p$ such vertices.

Vertices in $P_2 \cup P_4$ have degree p. There are $|P_2| + |P_4| = 1 + (p-1) = p$ such vertices.

The sum of all vertex degrees is:

SUM_$\{v \in V(\Gamma)\} deg(v) = p \cdot (2p-1) + p \cdot p = p(2p-1+p) = p(3p-1)$

By the Handshaking Lemma, $\left|E\left(\Gamma_{\{Z_n,H\}}\right)\right| = \frac{1}{2} * SUM\ deg(v) = \frac{p(3p-1)}{2}$.

Example 3.2. Let $p = 7$, so $Z_\{14\} = \{0,1,\ldots,13\}$.

$P_1 = \{0\}$, $P2 = \{7\}$, $P_3 = \{2,4,6,8,10,12\}$ $(|P_3| = 6)$, $P_4 = \{1,3,5,9,11,13\}(|P_4| = 6)$ *By Theorem* 3.4: $\left|E\left(\Gamma_{\{Z_n,H\}}\right)\right| = \frac{7(3 \cdot 7-1)}{2} = \frac{7(21-1)}{2} = \frac{7 \times 20}{2} = 70$ edges.

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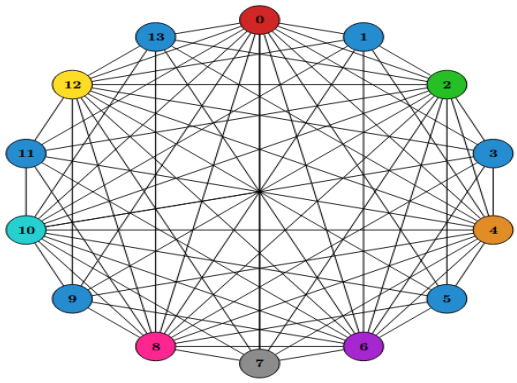

Figure 2: Relatively Prime Coprime Graph of Z_{14} $(p = 7)$

The relative prime coprime graph for Z_{14}, illustrating the degree distribution and edge count.

Theorem 3.5 (Non-Completeness of $\Gamma_{\{Z_n,H\}}$). Let $p > 2$ be a prime and let $\Gamma_{\{Z_n,H\}}$ be the relative prime coprime graph of Z_{2p}. Then $\Gamma_{\{Z_n,H\}}$ is not a complete graph.

Proof

Consider any two distinct vertices $a, b \in P_4$. By Lemma 3.1, $o(a) = o(b) = 2p$. Therefore, $gcd(o(a), o(b)) = gcd(2p, 2p) = 2p$. The adjacency criterion requires $gcd = 1$ *or* $gcd = p$. Since $2p \notin \{1, p\}$, the vertices a and b are not adjacent. Thus, $\Gamma_{\{Z_n,H\}}$ omits at least all edges between distinct vertices in P_4 and is not complete.

Example 3.3. Let $p = 3, so\ Z_6 = \{0,1,2,3,4,5\}$, $P_1 = \{0\}$, $P_2 = \{3\}$, $P_3 = \{2,4\}$, $P_4 = \{1,5\}$. Consider vertices $1,5 \in P_4$. Their orders are $o(1) = o(5) = 6$. Then $gcd(6,6) = 6$, which is neither 1 nor 3. Therefore, 1 and 5 are not adjacent, proving $\Gamma_{\{Z_n,H\}}$ is not complete.

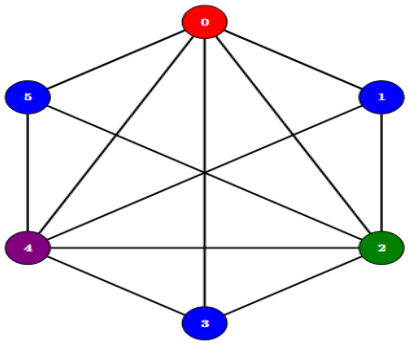

Figure 3: Relatively Prime Coprime Graph of Z_6 $(p = 3)$

The non-complete relative prime coprime graph for Z_6. The missing edge between vertices 1 and 5 is highlighted.

4. DISCUSSION

Based on the theoretical results presented, the relative prime coprime graph $\Gamma_{\{Z_n,H\}}$ on the cyclic group Z_{2p} exhibits a remarkably clear structure. The research establishes that the vertex set partitions into four distinct subsets, leading to the definitive result that elements of orders 1 and p act as universal vertices with degree $2p - 1$, while elements of orders 2 and $2p$ have a significantly lower degree of p, creating a precise and predictable connectivity pattern. This inherent structure directly facilitates the derivation of an exact combinatorial formula for the graph's size, $\left|E\left(\Gamma_{\{Z_n,H\}}\right)\right| = \frac{p(3p-1)}{2}$, and provides a proof of the graph's non-completeness, as vertices of order $2p$ are never adjacent to each other. These findings demonstrate that a simple adjacency rule based on greatest common divisors of element orders generates a complex graph with highly organized properties, for a strong analytically connection between the algebraic structure of the group and the resulting graph.

5. CONCLUSION

The research successfully characterizes the relative prime coprime graph on Z_{2p} with mathematical precision, revealing elegant structural patterns governed by elementary number theory. The clear dichotomy in vertex degrees and exact combinatorial formulas provide both theoretical insight and practical utility for applications in network science and computer science. The results demonstrate that even simple adjacency rules based on number-theoretic properties can generate rich, structured graphs with predictable and analysable properties.

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