DOI:https://doi.org/10.57233/ihcfs.041 ISBN: 978-978-68-07775

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1 ST INTERNATIONAL HYBRID CONFERENCE OF FACULTY OF SCIENCE

(A PUBLICATION OF FACULTY OF SCIENCE, FEDERAL UNIVERSITY GUSAU, NIGERIA)

ON A SPECIFIC SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY MODIFIED AL-OBOUDI DIFFERENTIAL OPERATOR

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ABSTRACT

This study presents and examines a novel subclass of analytic and univalent functions within a unit disc, employing the modified Al-oboudi differential operator. The aim of this research is to acertain some geometric properties, namely, the inclusion, closure and coefficient estimate for this class.

Keywords: Al-oboudi differential operator, Fekete-Szego functional, coefficient estimate, analytic, univalent**.**

1.0 INTRODUCTION

Operators are fundamental in mathematics, and play a crucial role in various branches of mathematics especially in geometric functions theory. Therefore, the investigation of operators is important, particularly in geometric function theory. Because of their importance in generating and preserving subclasses of regular univalent functions, operators have attracted a lot of attention in the geometric functions theory. Many mathematicians have written on operators and the new generalizations in a range of articles. Since 1915, operators have been the subject of research. Since then, mathematicians are still very interested in discussing operators with many properties of regular univalent functions. In this work, a class of univalent functions $B_n(\alpha,\beta,\lambda)$ formulated by a new generalized differential operator in the open unit disk.

The class of functions f(z) that satisfy the conditions f(0) = 0, f'(0) = 1, will be denoted by A. Such functions are written in the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

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Definition 1. Let $f \in A$, then we define the new operator as follow:

$$
\begin{aligned}
&D^0 f(z)^\alpha = f(z)^\alpha \\
&D^1_{\lambda,\alpha} f(z)^\alpha = \alpha(1-\lambda) f(z)^\alpha + \lambda z (f(z)^\alpha)' = D_{\lambda,\alpha} f(z)^\alpha, \qquad \lambda \geq 0, \quad \alpha > 0 \\
&D^n_{\lambda,\alpha} f(z)^\alpha = \alpha(1-\lambda) D^{n-1}_{\lambda,\alpha} f(z)^\alpha + \lambda z \left[D^{n-1}_{\lambda,\alpha} f(z)^\alpha \right]' = D_{\lambda,\alpha} \left[D^{n-1}_{\lambda,\alpha} f(z)^\alpha \right] \\
&D^n_{\lambda,\alpha} f(z)^\alpha = \alpha^n z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) \left[\alpha + \lambda(k-1) \right]^n z^{\alpha+k-1}
\end{aligned}
\tag{1}
$$

Definition 2. A normalized regular function $f(z)$ is said to be in $B_n(\alpha, \lambda, \beta)$ if and only if

$$
Re \frac{D^n_{\lambda,\alpha} f(z)^\alpha}{\alpha^n z^\alpha} > \beta \tag{2}
$$

for non-negative real number α, $0 < \lambda \leq 1$ and β $(0 \leq \beta < 1)$, where $D^n_{\lambda,\alpha}$, $(n \in N_0)$is as already defined.

The class $B_n(\alpha, \lambda, \beta)$ serves as a new generalization of many known subclasses of analytic functions in the direction by means of various choice of the parameters involved. For instance, several subclasses of functions are given in the following remarks.

Remark 1. If we put $\lambda = 1$, we have the class of functions by studied Babalola and Opoola (2006).

Remark 2. If we put $\lambda = 1, \beta = 0$, we have the class of functions by studied Abdulhalim (2003).

Remark 3. If we put $\lambda = 1, \beta = 0, n = 1$, we have the class of functions by studied Bazilevic(1955) and Singh (1973).

Remark 4. If we put $\alpha = 1, \lambda = 1, \beta = 0, n = 1$, we have the class of functions by studied Yamaguchi (1966).

2.0 Preliminary Lemmas

Let P denote the class of functions $P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ which are regular in unit disk, E and satisfy $Rep(z) > 0$ for any $z \in E$. To prove the

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main results the following lemmas would be needed.

Lemma 1. Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Define Ψ as the set of functions $\psi(u, v) : C \times C \to C$ satisfying:

(a) $\psi(u, v)$ is continuous in a domain Ω of $C \times C$,

(b) $(1, 0) \in \Omega$ and $\text{Re}\psi(1, 0) > 0$,

(c) $Re\psi(u_2 i, v_1) \leq 0$ when $(u_2 i, v_1) \in \Omega$ and $2v_1 \leq -(1 + u_2^2)$.

Babalola and Opoola (2006)

Lemma 2. Let $p \in P$. Then $|c_k| \leq 2$, $k = 1, 2, 3, \ldots$, and the inequality is sharp.

Lemma 3. Let $p \in P$. Then

$$\left| p_2 - \sigma \frac{p_1^2}{2} \right| \leq \begin{cases} 2(1-\sigma) & if \sigma \leq 0 \\ 2 & if 0 \leq \sigma \leq 2, = 2max\left\{1, |1-\sigma|\right\}. \\ 2(\sigma - 1), & if \sigma \geq 2 \end{cases}$$

Ibrahim (2014).

3.0 The Main Results

Theorem 1. $B_{n+1}(\alpha, \lambda, \beta) \subset B_n(\alpha, \lambda, \beta)$

Proof. Let $\frac{D^n_{\lambda,\alpha} f(z)^\alpha}{\alpha^n z^\alpha} = p(z)$

Since,

$$D^{n+1}_{\lambda,\alpha} f(z)^\alpha = \alpha(1-\lambda) D^n_{\lambda,\alpha} f(z)^\alpha + \lambda z \left[D^n_{\lambda,\alpha} f(z)^\alpha \right]'$$

By further simplification, we have

$D^{n+1}_{\lambda,\alpha} f(z)^\alpha = \alpha^{n+1} z^\alpha p(z) + \lambda \alpha^n z^{\alpha+1} p'(z)$

$$\frac{D^{n+1}_{\lambda,\alpha} f(z)^\alpha}{\alpha^{n+1} z^\alpha} = p(z) + \frac{\lambda z p'(z)}{\alpha}$$

implies that if $f \in B_{n+1}(\alpha, \lambda, \beta)$, then

$$Re \frac{D^{n+1}_{\lambda,\alpha} f(z)^\alpha}{\alpha^{n+1} z^\alpha} = Re\left(p(z) + \frac{z p'(z)}{\alpha \lambda^{-1}} \right) > \beta$$

Now define $\psi(u, v) = u + \frac{v}{\alpha \lambda^{-1}}$ on a domain. Also $\alpha \lambda^{-1} > 0$ and real, then ψ satisfies all the conditions in Lemma 1, It follows that

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$\psi(u,v)$ is continuous, $\psi \in \psi(u,v)$ and $Re\psi(1,0) > 1$. $Re\psi(\beta+(1-\beta)u_2 i, v_1) = \beta + \frac{v_1}{\alpha\lambda^{-1}}$ where $v_1 \leq \frac{(1-\beta)(1+u_2^2)}{2}$.

$$Re\frac{D^n_{\lambda,\alpha}f(z)^\alpha}{\alpha^n z^\alpha} = Rep(z) > \beta$$

That is

$$f \in B_n(\alpha,\lambda,\beta)$$

Theorem 2. The class $B_n(\alpha,\lambda,\beta)$ is closed under the integral

$$G(z)^\alpha = \frac{\alpha+\gamma}{\alpha z^\gamma}\int_0^z t^{\gamma-1}f(t)^\alpha dt,$$

where

$$\gamma = \frac{\alpha}{\lambda} - \alpha.$$

Proof. From integral we have

$$\frac{\alpha(z^\gamma G(z)^\alpha)'}{\gamma+\alpha} = z^{\gamma-1}f(z)^\alpha$$

then,

$$\frac{\alpha(\gamma z^{\gamma-1}G(z)^\alpha + z^\gamma(G(z)^\alpha)')}{\gamma+\alpha} = z^{\gamma-1}f(z)^\alpha$$
$$(\alpha-\alpha\lambda)G(z)^\alpha + z\lambda(G(z)^\alpha)' = f(z)^\alpha$$

this implies,

$$D_{\lambda,\alpha}G(z)^\alpha = f(z)^\alpha$$
$$\frac{D^{n+1}_{\lambda,\alpha}G(z)^\alpha}{\alpha^n z^\alpha} = \frac{D^n_{\lambda,\alpha}f(z)^\alpha}{\alpha^n z^\alpha}$$

Define

$$\frac{D^n_{\lambda,\alpha}G(z)^\alpha}{\alpha^n z^\alpha} = p(z),$$

then,

$$\frac{D^{n+1}_{\lambda,\alpha}G(z)^\alpha}{\alpha^n z^\alpha} = \alpha p(z) + \lambda z p'(z)$$

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so that

$$\frac{D^n_{\lambda,\alpha} f(z)^\alpha}{\alpha^n z^\alpha} = \alpha p(z) + \lambda z p'(z)$$

$$Re\frac{D^n_{\lambda,\alpha} f(z)^\alpha}{\alpha^n z^\alpha} = Re\alpha\left(p(z) + \frac{\lambda z p'(z)}{\alpha}\right) > \beta$$

since $f \in B^\alpha_n(\lambda, \beta)$. By Lemma 1, $Rep(z) > \beta$.

It follows that

$$Re\frac{D^n_{\lambda,\alpha} f(z)^\alpha}{\alpha^n z^\alpha} = Re\alpha\left(p(z) + \frac{\lambda z p'(z)}{\alpha}\right) = \left(\beta + \frac{v_1}{\alpha\lambda^{-1}}\right) > \beta$$

which complete the proof.

Theorem 3. Let $f \in B_n(\alpha, \lambda, \beta)$. Then

$$|a_2| \leq \frac{2\alpha^{n-1}}{(\alpha+\lambda)^n},$$

$$|a_3| \leq \begin{cases} \left\{\frac{2\alpha^{n-1}}{(\alpha+2\lambda)^n}\left[1 - \frac{(\alpha-1)(\alpha^2+2\lambda\alpha)^n}{2\alpha(\alpha+\lambda)^{2n}}\right], & for \quad 0 < \alpha < 1, \\ \frac{2\alpha^{n-1}}{(\alpha+2\lambda)^n} & for \quad \alpha \geq 1 \end{cases} \tag{3}$$

and

$$|a_4| \leq \frac{2\alpha^{n-1}}{(\alpha+3\lambda)^n}\left[\begin{array}{c} 1 + \frac{2(1-\alpha)(\alpha+3\lambda)^n\alpha^{n-1}}{(\alpha+\lambda)^n} \\ \left[\frac{1}{(\alpha+2\lambda)^n} + \frac{(1-2\alpha)\alpha^{n-1}}{3(\alpha+\lambda)^{2n}}\right] \end{array}\right]$$

Proof

Let $f \in B_n(\alpha, \lambda, \beta)$, then there exists $p \in P(z)$ such that

$$D^n_{\lambda,\alpha} f(z)^\alpha = \alpha^n z^\alpha p(z).$$

The series expansion of the left-hand side of the above equation is

$$D^n_{\lambda,\alpha} f(z)^\alpha = \alpha^n z^\alpha + \alpha a_2 [\alpha+\lambda]^n z^{\alpha+1} + \left(\alpha a_3 + \frac{\alpha(\alpha-1)a_2^2}{2}\right)[\alpha+2\lambda]^n z^{\alpha+2}$$

$$+ \left(\alpha a_4 + \alpha(\alpha-1)a_2 a_3 + \frac{(\alpha^3 - 3\alpha^2 + 2\alpha)a_2^3}{6}\right)[\alpha+3\lambda]^n z^{\alpha+3}$$

$$= \alpha^n z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha)[\alpha+\lambda(k-1)]^n z^{\alpha+k-1}. \tag{4}$$

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And that of the right hand side is

$$\alpha^n z^\alpha p(z) = \alpha^n z^\alpha + \alpha^n c_1 z^{\alpha+1} + \alpha^n c_2 z^{\alpha+2} + \alpha^n c_3 z^{\alpha+3} + \alpha^n c_4 z^{\alpha+4} + \cdots$$

Comparing the coefficients, we have

$$\alpha a_2 \left[\alpha + \lambda\right]^n = \alpha^n c_1$$

$$\left(\alpha a_3 + \frac{\alpha(\alpha-1)a_2^2}{2}\right)\left[\alpha + 2\lambda\right]^n = \alpha^n c_2$$

$$\left(\alpha a_4 + \alpha(\alpha-1)a_2 a_3 + \frac{\alpha(\alpha-1)(\alpha-2)a_2^3}{6}\right)\left[\alpha + 3\lambda\right]^n = \alpha^n c_3.$$

To obtain the bound on a_2,

$$a_2 = \frac{\alpha^{n-1} c_1}{(\alpha+\lambda)^n}$$

Taking modulus of both sides and applying Lemma 2, gives the bound on a_2,

$$|a_2| = \frac{2\alpha^{n-1}}{(\alpha+\lambda)^n}.$$

Next is the bound on a_3.

$$\left(\alpha a_3 + \frac{\alpha(\alpha-1)a_2^2}{2}\right)\left[\alpha + 2\lambda\right]^n = \alpha^n c_2$$

then,

$$a_3 = \frac{\alpha^{n-1} c_2}{\left[\alpha + 2\lambda\right]^n} + \frac{(1-\alpha)a_2^2}{2}$$

Substituting the value of a_2, we have

$$a_3 = \frac{\alpha^{n-1} c_2}{\left[\alpha + 2\lambda\right]^n} + \frac{(1-\alpha)}{2}\left(\frac{c_1 \alpha^{n-1}}{(\alpha+\lambda)^n}\right)^2$$

$$a_3 = \frac{\alpha^{n-1}}{\left[\alpha + 2\lambda\right]^n}\left[c_2 - \frac{(\alpha-1)(\alpha+2\lambda)^n \alpha^{n-1} c_1^2}{2(\alpha+\lambda)^{2n}}\right]$$

$$a_3 = \frac{\alpha^{n-1}}{\left[\alpha + 2\lambda\right]^n}\left[c_2 - \chi c_1^2\right]$$

$$|a_3| = \frac{\alpha^{n-1}}{\left[\alpha + 2\lambda\right]^n}\left|c_2 - \chi c_1^2\right|$$

Applying Lemma 2 and 3, we have

$$|a_3| = \frac{2\alpha^{n-1}}{\left[\alpha + 2\lambda\right]^n}\max\{1, |1-2\chi|\} \qquad \text{for any} \quad \chi \in C,$$

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where

$$2\chi = \frac{(\alpha-1)(\alpha+2\lambda)^n\alpha^{n-1}c_1^2}{(\alpha+\lambda)^{2n}}$$

To obtain a_4

$$a_4 = \frac{\alpha^{n-1}c_3}{[\alpha+3\lambda]^n} + (1-\alpha)a_2a_3 + \frac{(1-\alpha)(\alpha-2)a_2^3}{6}$$

Substituting a_2 and a_3, we have

$$a_4 = \frac{\alpha^{n-1}c_3}{[\alpha+3\lambda]^n} + (1-\alpha)\left(\frac{\alpha^{n-1}c_1}{(\alpha+\lambda)^n}\right)\left(\frac{\alpha^{n-1}c_2}{[\alpha+2\lambda]^n} + \frac{(1-\alpha)}{2}\left(\frac{c_1\alpha^{n-1}}{(\alpha+\lambda)^n}\right)^2\right)$$

$$+\frac{(1-\alpha)(\alpha-2)\left(\frac{\alpha^{n-1}c_1}{(\alpha+\lambda)^n}\right)^3}{6}$$

$$= \frac{\alpha^{n-1}}{(\alpha+3\lambda)^n}\left[c_3 + \frac{(1-\alpha)(\alpha-2)(\alpha+3\lambda)^n\alpha^{2(n-1)}c_1^3}{6(\alpha+\lambda)^{3n}} + \frac{(1-\alpha)(\alpha+3\lambda)^n\alpha^{n-1}c_1c_2}{(\alpha+\lambda)^n(\alpha+2\lambda)^n}\right] +$$

$$\frac{\alpha^{n-1}}{(\alpha+3\lambda)^n}\left[\frac{(1-\alpha)^2(\alpha+3\lambda)^n\alpha^{2(n-1)}c_1^3}{2(\alpha+\lambda)^{3n}}\right]$$

$$= \frac{\alpha^{n-1}}{(\alpha+3\lambda)^n}\left[\begin{array}{r} c_3 + \frac{(1-\alpha)(1-2\alpha)(\alpha+3\lambda)^n\alpha^{2(n-1)}c_1^3}{6(\alpha+\lambda)^{3n}} + \\ \frac{(1-\alpha)(\alpha+3\lambda)^n\alpha^{n-1}c_1c_2}{(\alpha+\lambda)^n(\alpha+2\lambda)^n} \end{array}\right]$$

$$= \frac{\alpha^{n-1}}{(\alpha+3\lambda)^n}\left[\begin{array}{l} c_3 + \frac{(1-\alpha)(\alpha+3\lambda)^n\alpha^{n-1}}{(\alpha+\lambda)^n} \\ \left[\frac{c_1c_2}{(\alpha+2\lambda)^n} + \frac{(1-2\alpha)\alpha^{n-1}c_1^3}{6(\alpha+\lambda)^{2n}}\right] \end{array}\right]$$

By application of Lemma 2,

$$|a_4| \leq \frac{2\alpha^{n-1}}{(\alpha+3\lambda)^n}\left[\begin{array}{l} 1 + \frac{2(1-\alpha)(\alpha+3\lambda)^n\alpha^{n-1}}{(\alpha+\lambda)^n} \\ \left[\frac{1}{(\alpha+2\lambda)^n} + \frac{(1-2\alpha)\alpha^{n-1}}{3(\alpha+\lambda)^{2n}}\right] \end{array}\right]$$

Theorem 4. Let $f \in B_n(\alpha,\lambda,\beta)$. Then

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{2\alpha^{n-1}}{(\alpha+2\lambda)^n}(1-\sigma) & if\sigma \leq 0 \\ \frac{2\alpha^{n-1}}{(\alpha+2\lambda)^n} & if 0 \leq \sigma \leq 2, = \frac{2\alpha^{n-1}}{(\alpha+2\lambda)^n} max\left\{1,|1-\sigma|\right\}. \\ \frac{2\alpha^{n-1}}{(\alpha+2\lambda)^n}(\sigma-1), & if\sigma \geq 2 \end{cases}$$

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where

$$\sigma = \frac{(\alpha+2\mu-1)(\alpha+2\lambda)^n\alpha^{n-1}}{(\alpha+\lambda)^{2n}}.$$

Proof. Let $f \in B_n(\alpha,\lambda,\beta)$. Then there exists a $p \in P$ such that $D^n_{\lambda,\alpha}f(z)^\alpha = \alpha^n z^\alpha p(z)$, on substituting a_2 and a_3, we have

$$\begin{aligned}
|a_3-\mu a_2^2| &= \left|\frac{\alpha^{n-1}}{(\alpha+2\lambda)^n}\left[c_2-\frac{(\alpha-1)(\alpha+2\lambda)^n\alpha^{n-1}c_1^2}{2(\alpha+\lambda)^{2n}}\right]-\mu\left(\frac{\alpha^{n-1}c_1}{(\alpha+\lambda)^n}\right)^2\right| \\
&= \left|\frac{\alpha^{n-1}c_2}{(\alpha+2\lambda)^n}-\frac{(\alpha+2\mu-1)\alpha^{2(n-1)}c_1^2}{2(\alpha+\lambda)^{2n}}\right| \\
&= \frac{\alpha^{n-1}}{(\alpha+2\lambda)^n}\left|c_2-\frac{(\alpha+2\mu-1)(\alpha+2\lambda)^n\alpha^{n-1}c_1^2}{2(\alpha+\lambda)^{2n}}\right| \\
&= \frac{\alpha^{n-1}}{(\alpha+2\lambda)^n}\left|c_2-\sigma\frac{c_1^2}{2}\right|
\end{aligned}$$

where

$$\sigma = \frac{(\alpha+2\mu-1)(\alpha+2\lambda)^n\alpha^{n-1}}{(\alpha+\lambda)^{2n}}.$$

By Lemma 3, the proof follows.

4.0 Conclusion

In the paper we defined a new differential operator. We used the new operator to study a new class of analytic and univalent functions, and also to obtain new results.

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